The stability of free boundary layers between two uniform streams

By T. TATSUMI AND K. GOTOH
Department of Physics, Faculty of Science, University of Kyoto, Japan

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Hydrodynamic stability of free boundary-layer flows is treated in general. It is found that the situations at low Reynolds numbers are universal for all velocity profiles of free boundary-layer type. Curves of constant amplification are calculated as far as $O(R^3)$. In particular, the asymptotic form of the neutral curves for $R \doteqdot 0$ is found to be $\alpha = R/(4\sqrt{3})$, so that the critical Reynolds numbers of these flows are identically zero. The phase velocity of the disturbance is also found to be zero, for all disturbances, up to the second approximation.

A method of normalizing the velocity profiles is suggested, and existing results for the stability of various profiles at large Reynolds numbers are discussed from a new point of view.

1. Introduction

It has recently become recognized that the problem of hydrodynamic stability for unbounded laminar flows is considerably different from that for bounded flows. For instance, it has been found that a plane laminar jet in an infinite fluid is highly unstable, and its critical Reynolds number is about 4 (Tatsumi & Kakutani 1958; Howard 1959) in contrast with the value 420 for the boundary layer along a flat plate and 6000 for plane Poiseuille flow. The mathematical behaviour of the eigen-solutions in the neighbourhood of the critical Reynolds number is also quite different for unbounded and bounded flows. In the latter, the solutions are sensitive to the second spatial derivative of the velocity profile, while in the former they may be expressed in terms of definite integrals of the profile, and so are largely independent of its detailed structure.

This difference seems to be particularly marked in the case of free boundary-layer flows between two uniform streams, since, according to Esch (1957), a simple piecewise linear profile of free boundary-layer type has zero critical Reynolds number, which means that the flow is always unstable. The stability of a realistic velocity profile of free boundary-layer type was first investigated by Lessen (1950). However, since he employed a method of analysis which is effective only for large Reynolds numbers, no result was given for the critical Reynolds number, which was supposed to be very low. Even in the region of large Reynolds number which was covered by the analyses of both Lessen and Esch, the behaviour of the respective curves of neutral stability is not identical. According to Lessen's result, the range of the wave-number α corresponding to instability decreases monotonically with decreasing Reynolds number R. On the other hand, Esch's neutral curve of $\alpha(R)$ shows a curious kink at about $R = 10$ giving there a widest wave-number range of instability. This result clearly contradicts the general belief regarding the stability of unbounded flows that viscosity always acts on the disturbance as a stabilizing factor.

In order to clarify these points, we attempt in this paper to discuss the stability of free boundary-layer flows in general, that is, assuming no particular form of the velocity profile. The mathematical method employed here is the same as that developed by Tatsumi & Kakutani for treating a plane laminar jet at low Reynolds numbers (Tatsumi & Kakutani 1958; this paper will be referred to hereafter as I).

The asymptotic behaviour of the neutral curve and the distribution of the amplification factor in the (α, R)-plane are obtained for small values of α and R, and they are found to be *universal* for all flows of this type, if the profiles are normalized with respect to some appropriate characteristic length. Among the universal properties of free boundary-layer flows at *small* Reynolds numbers, the following two may be noted: (i) the critical Reynolds numbers of these flows are zero, and (ii) the phase velocities of all disturbances are identically zero.

If the velocity profiles are normalized so as to make their stability properties at low Reynolds numbers universal, they generally show different behaviour for larger Reynolds numbers. The above-mentioned difference between the high-R branches of Lessen's and Esch's neutral curves is therefore easily accountable from this point of view. It may be concluded in general that we cannot use a rough approximation to a velocity profile for investigating its stability at larger Reynolds numbers.

2. Formulation of the problem

Let $U(y)$ be the velocity profile of a steady plane parallel flow, taking the x-axis of Cartesian co-ordinates along the direction of the flow. In stability problems of plane flow, Squire's theorem (1933) guarantees that we need only consider two-dimensional disturbances, provided they are smaller in order of magnitude than the velocity of the undisturbed flow.

Two-dimensional disturbance velocities (u, v) may be expressed in terms of the stream function $\psi(x, y, t)$ as

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \tag{2.1}$$

In particular, we consider here a harmonic component

$$\psi = \phi(y) \exp \{i\alpha(x-ct)\}, \tag{2.2}$$

where $\alpha\ (>0)$ represents the wave-number in the x-direction, $\mathscr{R}(c) \equiv c_r$ the phase velocity, and $\alpha\mathscr{I}(c) \equiv \alpha c_i$ the amplification factor of the disturbance. According as c_i takes positive, zero or negative values, we have an amplified, neutral or damped disturbance respectively.

Substituting (2.1) and (2.2) into the equations of motion and neglecting the non-linear terms with respect to ϕ, we obtain the Orr–Sommerfeld equation

$$\phi^{\text{iv}} - 2\alpha^2\phi'' + \alpha^4\phi = i\alpha R\{(U-c)(\phi''-\alpha^2\phi) - U''\phi\}, \tag{2.3}$$

where dashes denote the differentiation with respect to y, and all quantities are made non-dimensional using the characteristic velocity U_0, scale l, and Reynolds number $R = U_0 l/\nu$. In the following, U_0 is so chosen as to make

$$U(\infty) = 1, \quad U(-\infty) = -1, \tag{2.4}$$

and l will be specified later.

Now we divide the whole space into two regions (I) where $y > 0$, and (II) where $y < 0$, and define $U_j(y)$ ($j = \text{I, II}$) by

$$\left.\begin{aligned} U_{\text{I}}(y) &= U(y) - U(\infty) = U(y) - 1, \\ U_{\text{II}}(y) &= U(y) - U(-\infty) = U(y) + 1. \end{aligned}\right\} \tag{2.5}$$

Then equation (2.3) may be written, for the regions I and II, as

$$(D^2 - \alpha^2)(D^2 - \beta_j^2)\phi = i\alpha R\{U_j(D^2 - \alpha^2) - U_j''\}\phi, \tag{2.6}$$

where

$$D \equiv d/dy, \quad \beta_{\text{I}}^2 = \alpha^2 - i\alpha R(c-1), \quad \beta_{\text{II}}^2 = \alpha^2 - i\alpha R(c+1), \quad \mathscr{R}(\beta_j) > 0.$$

In the region (I), U_{I} and U_{I}'' on the right-hand side of equation (2.6) decrease with increasing y, and in the region (II) the same is true for U_{II} and U_{II}'' with increasing $-y$. Thus, the premises for expanding the solution ϕ in power series of αR, which was explored in § 5 of I, are satisfied in each region separately, and we can express ϕ_j in the form

$$\phi_j(y) = \sum_{n=0}^{\infty} (i\alpha R)^n \phi_j^{(n)}(y; \alpha, \beta_j), \tag{2.7}$$

where the ϕ_j's are the solutions of the following equations:

$$\left.\begin{aligned} &(D^2 - \alpha^2)(D^2 - \beta_j^2)\phi_j^{(0)} = 0, \\ &(D^2 - \alpha^2)(D^2 - \beta_j^2)\phi_j^{(n)} = \{U_j(D^2 - \alpha^2) - U_j''\}\phi_j^{(n-1)} \quad (n \geqslant 1). \end{aligned}\right\} \tag{2.8}$$

The uniform convergence of the series (2.7) can be easily verified, by extending the proof given in § 5 of I, for those velocity profiles in which U_{I} and U_{II} tend to zero as, or more rapidly than, $e^{\mp my}$ for $y \to \pm\infty$ respectively, and provided that $\alpha < m$, $\mathscr{R}(\beta_j) < m$. The above conditions for $U(y)$ are usually satisfied if the boundary-layer approximation is valid for the basic flow. For, in that case, the stream function, denoted by f, must satisfy

$$2f''' + ff'' = 0,$$

and asymptotically

$$U(y) = f'(y) \propto \frac{1}{y} e^{-my^2} + \text{const.}, \quad \text{or} \quad e^{\mp my} + \text{const.},$$

according as $f'(\pm\infty) [= U(\pm\infty)] \neq 0$ or $f'(\pm\infty) [= U(\pm\infty)] = 0$ respectively.

The boundary condition for the disturbance is that the component velocities must vanish at $y = \pm\infty$, that is,

$$\phi'(\pm\infty) = \alpha\phi(\pm\infty) = 0. \tag{2.9}$$

3. Equation for eigenvalues

Equations (2.8) permit four independent solutions for each ϕ_j, but two of them must always be rejected by the condition (2.9). Thus we have the solutions

$$\left.\begin{aligned}\phi_{\mathrm{I}} &= C_1\phi_{\mathrm{I}1}+C_2\phi_{\mathrm{I}2} \quad \text{for} \quad y>0,\\ \phi_{\mathrm{II}} &= C_3\phi_{\mathrm{II}1}+C_4\phi_{\mathrm{II}2} \quad \text{for} \quad y<0,\end{aligned}\right\} \tag{3.1}$$

where the C's are numerical constants, and $\phi_{jk}\,(j=\mathrm{I},\mathrm{II};\ k=1,2)$ are given by

$$\phi_{jk} = \sum_{n=0}^{\infty} (i\alpha R)^n\,\phi_{jk}^{(n)}(y;\alpha,\beta_j), \tag{3.2}$$

with

$$\phi_{\mathrm{I}1}^{(0)} = e^{-\alpha y}, \quad \phi_{\mathrm{I}2}^{(0)} = e^{-\beta_{\mathrm{I}} y}, \quad \phi_{\mathrm{II}1}^{(0)} = e^{\alpha y}, \quad \phi_{\mathrm{II}2}^{(0)} = e^{\beta_{\mathrm{II}} y}, \tag{3.3}$$

and

$$\begin{aligned}\phi_{jk}^{(n)} = \frac{1}{\alpha^2-\beta_j^2}\Bigg[&e^{-\alpha y}\int_{\pm\infty}^{y} U_j e^{\alpha y}(D+\alpha)\,\phi_{jk}^{(n-1)}\,dy + e^{\alpha y}\int_{\pm\infty}^{y} U_j e^{-\alpha y}(D-\alpha)\,\phi_{jk}^{(n-1)}\,dy\\ &-e^{-\beta_j y}\int_{\pm\infty}^{y} U_j e^{\beta_j y}\left(D+\frac{\alpha^2+\beta_j^2}{2\beta_j}\right)\phi_{jk}^{(n-1)}\,dy - e^{\beta_j y}\int_{\pm\infty}^{y} U_j e^{-\beta_j y}\left(D-\frac{\alpha^2+\beta_j^2}{2\beta_j}\right)\phi_{jk}^{(n-1)}\,dy\Bigg]\end{aligned} \tag{3.4}$$

$$(j=\mathrm{I},\mathrm{II}, \quad k=1,2, \quad n\geqslant 1),$$

where $\pm\infty$ in the integrals must be taken as $+\infty$ and $-\infty$ for $j=\mathrm{I}$ and II respectively.

In order to obtain the complete solution ϕ of (2.3) throughout $-\infty<y<\infty$, we have to connect ϕ_{I} and ϕ_{II} analytically at $y=0$. This is done by putting

$$\left.\begin{aligned}\phi_{\mathrm{I}}(0) = \phi_{\mathrm{II}}(0), \quad \phi_{\mathrm{I}}'(0) = \phi_{\mathrm{II}}'(0),\\ \phi_{\mathrm{I}}''(0) = \phi_{\mathrm{II}}''(0), \quad \phi_{\mathrm{I}}'''(0) = \phi_{\mathrm{II}}'''(0),\end{aligned}\right\} \tag{3.5}$$

equality of all other higher derivatives being then automatically satisfied through these conditions and equation (2.3). Substituting (3.1) into (3.5), we have the following condition for all C's not to vanish:

$$E \equiv \begin{vmatrix}\phi_{\mathrm{I}1}(0) & \phi_{\mathrm{I}2}(0) & \phi_{\mathrm{II}1}(0) & \phi_{\mathrm{II}2}(0)\\ \phi_{\mathrm{I}1}'(0) & \phi_{\mathrm{I}2}'(0) & \phi_{\mathrm{II}1}'(0) & \phi_{\mathrm{II}2}'(0)\\ \phi_{\mathrm{I}1}''(0) & \phi_{\mathrm{I}2}''(0) & \phi_{\mathrm{II}1}''(0) & \phi_{\mathrm{II}2}''(0)\\ \phi_{\mathrm{I}1}'''(0) & \phi_{\mathrm{I}2}'''(0) & \phi_{\mathrm{II}1}'''(0) & \phi_{\mathrm{II}2}'''(0)\end{vmatrix} = 0. \tag{3.6}$$

This equation gives a relationship between eigenvalues of α, R and c. For real values of c it defines α as a function of R, which may be shown graphically in the (α, R)-plane as the neutral curve.

4. Eigenvalue problem for small α and R

Equation (3.6), with (3.2), (3.3) and (3.4) substituted, can be expanded into powers of $i\alpha R$ as follows:

$$E \equiv \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} (i\alpha R)^{m+n+p+q}\,E_{mnpq} = 0, \tag{4.1}$$

where

$$E_{mnpq} = \begin{vmatrix} I_{\mathrm{I}1}^{(m)} - J_{\mathrm{I}1}^{(m)} & I_{\mathrm{I}2}^{(n)} - J_{\mathrm{I}2}^{(n)} & I_{\mathrm{II}1}^{(p)} - J_{\mathrm{II}1}^{(p)} & I_{\mathrm{II}2}^{(q)} - J_{\mathrm{II}2}^{(q)} \\ \alpha M_{\mathrm{I}1}^{(m)} - \beta_{\mathrm{I}} N_{\mathrm{I}1}^{(m)} & \alpha M_{\mathrm{I}2}^{(n)} - \beta_{\mathrm{I}} N_{\mathrm{I}2}^{(n)} & \alpha M_{\mathrm{II}1}^{(p)} - \beta_{\mathrm{II}} N_{\mathrm{II}1}^{(p)} & \alpha M_{\mathrm{II}2}^{(q)} - \beta_{\mathrm{II}} N_{\mathrm{II}2}^{(q)} \\ (\alpha^2-\beta_{\mathrm{I}}^2) J_{\mathrm{I}1}^{(m)} & (\alpha^2-\beta_{\mathrm{I}}^2) J_{\mathrm{I}2}^{(n)} & (\alpha^2-\beta_{\mathrm{II}}^2) J_{\mathrm{II}1}^{(p)} - 2i\alpha R(I_{\mathrm{II}1}^{(p)} - J_{\mathrm{II}1}^{(p)}) & (\alpha^2-\beta_{\mathrm{II}}^2) J_{\mathrm{II}2}^{(q)} - 2i\alpha R(I_{\mathrm{II}2}^{(q)} - J_{\mathrm{II}2}^{(q)}) \\ \beta_{\mathrm{I}}(\alpha^2-\beta_{\mathrm{I}}^2) N_{\mathrm{I}1}^{(m)} & \beta_{\mathrm{I}}(\alpha^2-\beta_{\mathrm{I}}^2) N_{\mathrm{I}2}^{(n)} & \beta_{\mathrm{II}}(\alpha^2-\beta_{\mathrm{II}}^2) N_{\mathrm{II}1}^{(p)} + 2i\alpha R(\alpha M_{\mathrm{II}1}^{(p)} - \beta_{\mathrm{II}} N_{\mathrm{II}1}^{(p)}) & \beta_{\mathrm{II}}(\alpha^2-\beta_{\mathrm{II}}^2) N_{\mathrm{II}2}^{(q)} + 2i\alpha R(\alpha M_{\mathrm{II}2}^{(q)} - \beta_{\mathrm{II}} N_{\mathrm{II}2}^{(q)}) \end{vmatrix} \quad (4.2)$$

with

$$\left.\begin{aligned} I_{j1}^{(0)} &= 1, & I_{j2}^{(0)} &= 0, & J_{j1}^{(0)} &= 0, & J_{j2}^{(0)} &= -1, \\ M_{j1}^{(0)} &= (-1)^j, & M_{j2}^{(0)} &= 0, & N_{j1}^{(0)} &= 0, & N_{j2}^{(0)} &= (-1)^{j+1}, \end{aligned}\right\} \quad (4.3)$$

and

$$\left.\begin{aligned} I_{jk}^{(n)} &= \frac{2}{\beta_j^2-\alpha^2}\int_0^{\pm\infty} U_j(y)\{\cosh(\alpha y)D + \alpha\sinh(\alpha y)\}\phi_{jk}^{(n-1)}(y)\,dy, \\ J_{jk}^{(n)} &= \frac{2}{\beta_j^2-\alpha^2}\int_0^{\pm\infty} U_j(y)\left\{\cosh(\beta_j y)D + \frac{\alpha^2+\beta_j^2}{2\beta_j}\sinh(\beta_j y)\right\}\phi_{jk}^{(n-1)}(y)\,dy, \\ M_{jk}^{(n)} &= \frac{-2}{\beta_j^2-\alpha^2}\int_0^{\pm\infty} U_j(y)\{\sinh(\alpha y)D + \alpha\cosh(\alpha y)\}\phi_{jk}^{(n-1)}(y)\,dy, \\ N_{jk}^{(n)} &= \frac{-2}{\beta_j^2-\alpha^2}\int_0^{\pm\infty} U_j(y)\left\{\sinh(\beta_j y)D + \frac{\alpha^2+\beta_j^2}{2\beta_j}\cosh(\beta_j y)\right\}\phi_{jk}^{(n-1)}(y)\,dy \quad (n \geqslant 1). \end{aligned}\right\} \quad (4.4)$$

In principle we can calculate $\alpha(R, c)$ for any value of R by solving equation (4.1), so far as the conditions $\alpha < m$, $\mathscr{R}\{\beta_j\} < m$ $(j = \mathrm{I}, \mathrm{II})$ are satisfied. In practice however, the computation becomes laborious for large R, for then we have to take more and more terms in order to obtain accurate solutions. Here we shall restrict ourselves to the study of equation (4.1) in the region of small α and R. Then, expanding E_{mnpq} again into powers of α and β_j, and retaining only first few terms, we obtain the following equation for eigenvalues:

$$\begin{aligned} E = \{1 + \tfrac{1}{2}(\beta_{\mathrm{I}}-\beta_{\mathrm{II}})(V_{\mathrm{I}}-V_{\mathrm{II}}) + \tfrac{1}{8}(\beta_{\mathrm{I}}-\beta_{\mathrm{II}})^2(V_{\mathrm{I}}-V_{\mathrm{II}})^2\} \\ \times\{2\alpha(\alpha-\beta_{\mathrm{I}})(\beta_{\mathrm{II}}-\alpha)(\beta_{\mathrm{I}}+\beta_{\mathrm{II}})\}F = 0, \end{aligned} \quad (4.5)$$

with

$$\left.\begin{aligned} F &= \{\beta_{\mathrm{I}}^2+\beta_{\mathrm{II}}^2-\beta_{\mathrm{I}}\beta_{\mathrm{II}}+\alpha(\beta_{\mathrm{I}}+\beta_{\mathrm{II}})+\alpha^2\} \\ &\quad -\tfrac{1}{2}(\beta_{\mathrm{I}}-\beta_{\mathrm{II}})^2[\{\beta_{\mathrm{I}}^2+\beta_{\mathrm{II}}^2-\beta_{\mathrm{I}}\beta_{\mathrm{II}}+\alpha(\beta_{\mathrm{I}}+\beta_{\mathrm{II}})+\alpha^2\} \\ &\quad +(\beta_{\mathrm{I}}+\beta_{\mathrm{II}})\,(\beta_{\mathrm{I}}+\beta_{\mathrm{II}}-2\alpha)][W_{\mathrm{I}}-W_{\mathrm{II}}+\tfrac{1}{4}(V_{\mathrm{I}}-V_{\mathrm{II}})^2]+O(\alpha^5,\beta_j^5), \\ V_{\mathrm{I}} &= \int_0^{\infty} U_{\mathrm{I}}\,dy, \qquad V_{\mathrm{II}} = \int_0^{-\infty} U_{\mathrm{II}}\,dy, \\ W_{\mathrm{I}} &= \int_0^{\infty} yU_{\mathrm{I}}\,dy, \quad W_{\mathrm{II}} = \int_0^{-\infty} yU_{\mathrm{II}}\,dy, \end{aligned}\right\} \quad (4.6)$$

where use is made of the relationship $\beta_{\mathrm{I}}^2-\beta_{\mathrm{II}}^2 = 2i\alpha R$.

The first factor of the right-hand side of (4.5) comes from arbitrariness in the choice of the origin of y, and does not affect the eigenvalue problem. In fact, if we choose $y = 0$ so as to make $V_{\mathrm{I}} = V_{\mathrm{II}}$, this factor becomes unity. The second factor, put equal to zero, gives only a trivial solution: $R = 0$ and $\alpha = 0$. Thus the true eigenvalue equation must be

$$F = 0, \tag{4.7}$$

in which it is confirmed, as a matter of course, that the factor

$$[W_{\mathrm{I}} - W_{\mathrm{II}} + \tfrac{1}{4}(V_{\mathrm{I}} - V_{\mathrm{II}})^2]$$

is independent of the choice of the origin of y.

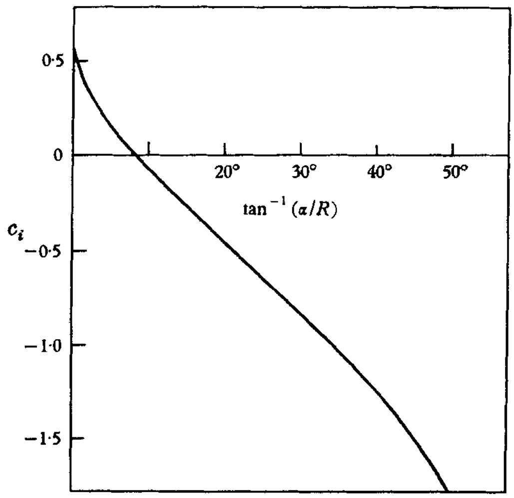

FIGURE 1. Distribution of c_i for small α and R.

As the first approximation, we take the equation

$$\beta_{\mathrm{I}}^2 + \beta_{\mathrm{II}}^2 - \beta_{\mathrm{I}}\beta_{\mathrm{II}} + \alpha(\beta_{\mathrm{I}} + \beta_{\mathrm{II}}) + \alpha^2 = 0, \tag{4.8}$$

the solution of which is given, under the conditions $\alpha > 0$, $\alpha R > 0$, $\mathscr{R}\{\beta_j\} > 0$ $(j = \mathrm{I}, \mathrm{II})$, by

$$\left.\begin{aligned} \alpha &= \frac{(1-\sqrt{3}\,c_i)^2}{4(\sqrt{3}-c_i)}\,R, \\ c_r &= 0. \end{aligned}\right\} \tag{4.9}$$

From (4.9) we obtain the distribution of the amplification factor c_i over the (α, R)-plane as shown in figure 1. Putting $c_i = 0$ in (4.9), we have the asymptotic neutral curve for small R

$$\alpha = \frac{R}{4\sqrt{3}}, \tag{4.10}$$

which agrees with the result obtained by Esch (1957) for a piecewise linear profile. It should be noted that the results (4.9) and (4.10) apply universally to all velocity profiles of the free boundary-layer type, that is, those profiles in which $U(\infty)$ and $U(-\infty)$ have a finite difference.

The second-approximate solution of (4.7) may be obtained by perturbation of the first approximation as

$$\left.\begin{aligned} \alpha &= \frac{(1-\sqrt{3}\,c_i)^2}{4(\sqrt{3}-c_i)}\,R\left\{1-\frac{(1-3c_i^2)\,(7-2\sqrt{3}\,c_i+c_i^2)}{4\sqrt{3}(\sqrt{3}-c_i)^3}\,(W_{\rm I}-W_{\rm II})\,R^2+O(R^3)\right\},\\ c_r &= 0. \end{aligned}\right\} \tag{4.11}$$

Thus, for the neutral curve,

$$\alpha = \frac{R}{4\sqrt{3}}\{1-\tfrac{7}{36}(W_{\rm I}-W_{\rm II})\,R^2+O(R^3)\}. \tag{4.12}$$

5. Discussion

It may be seen from (4.11) that the velocity profile $U(y)$ enters the problem only through the factor $W_{\rm I}-W_{\rm II}$. As we have left the characteristic length l unspecified, we have now the liberty of choosing l so as to make

$$W_{\rm II}-W_{\rm I} = 1. \tag{5.1}$$

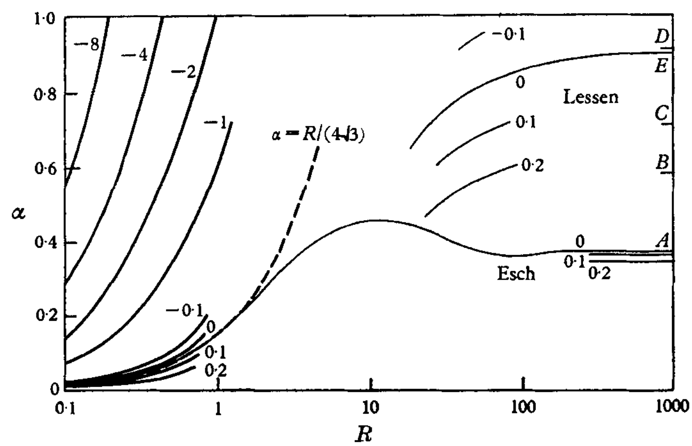

FIGURE 2. Curve of constant c_i. Numbers attached to the curves denote values of c_i.

With this choice of the characteristic length, the asymptotic behaviour of the curves of constant c_i becomes universal up to $O(R^3)$. These curves (including the neutral curve) calculated from (4.11) and (4.12) are shown in figure 2.

Figure 2 also shows the results obtained by Lessen (1950) and Esch (1957) for the asymptotic branch of the neutral curve at higher Reynolds numbers, their velocity profiles being normalized as to satisfy the condition (5.1). As is clearly seen from figure 2, the solutions cease to be universal for Reynolds numbers which are not small, and the neutral curves have different limiting values of wave-number, α_s say, for infinite Reynolds number. It is of course possible to choose l so as to make the high Reynolds number branches of any two neutral curves coincide, but then the situation at small Reynolds numbers is no longer unique. Since the condition (5.1) gives an absolute rule for determining l which is valid not only for a given pair of profiles but also for all possible profiles of free boundary-layer type, it seems most meaningful to adopt this condition for normalizing the velocity profiles.

As the neutral curves of Lessen's and Esch's profiles show a common feature that α does not change appreciably with R for larger values of R, the same trend may be expected to hold for other profiles, and high Reynolds number branches of their neutral curves may be well approximated by α = const. $= \alpha_s$. Thus it may be concluded that if we want to investigate the stability problem of a free boundary-layer flow using some approximate velocity profile, the profile must be such that it gives α_s close to that of the exact profile. The limiting wave-number α_s has been calculated for a number of typical velocity profiles, and its numerical values are tabulated in table 1 and marked in figure 2. It may be seen from table 1 that the profile $U(y) = \tanh ky$ gives the closest approximation to Lessen's profile which was obtained by solving the boundary-layer equation numerically.

	Antisymmetric profile	α_s
A	$\begin{cases} 0 < ky < 1,\ U = ky; \\ 1 < ky,\ U = 1;\ k = 1/\surd 3 \end{cases}$	0·37
B	$\begin{cases} 0 < 2ky < 1,\ U = ky; \\ 1 < 2ky < 3,\ U = (2ky+1)/4; \\ 3 < 2ky,\ U = 1;\ k = \surd 7/4 \end{cases}$	0·58
C	$U = 2\operatorname{erf} ky - 1,\ k = 1/\surd 2$	0·71
D	$U = \tanh ky,\ k = \pi/2\surd 3$	0·91
	Asymmetric profile	
E	Numerical solution (Lessen)	0·90

TABLE 1

Equation (4.11) also shows that the relation $c_r = 0$ holds up to the second approximation. In a general co-ordinate system which is free from the condition (2.4), this relation may be expressed as

$$c_r = \tfrac{1}{2}\{U(\infty) + U(-\infty)\}. \tag{5.2}$$

It is unlikely, however, that (5.2) is universally valid for all Reynolds numbers and profiles, and if not it must fail to be satisfied at some higher stage of approximation. If the velocity profile is antisymmetric with respect to y, $c_r = 0$ follows from the uniqueness of ϕ for given α and R. Since $U(-y) = -U(y)$ for this profile, we have, replacing y by $-y$ in the Orr–Sommerfeld equation (2.3),

$$\phi^{\text{iv}}(-y) - 2\alpha^2\phi''(-y) + \alpha^4\phi(-y) \\ = -i\alpha R[(U+c)\{\phi''(-y) - \alpha^2\phi(-y)\} - U''\phi(-y)]. \tag{5.3}$$

If we put $\chi(y) \equiv \bar{\phi}(-y)$, the equation for $\chi(y)$ is written, taking the complex conjugate of (5.3), as

$$\chi^{\text{iv}} - 2\alpha^2\chi'' + \alpha^4\chi = i\alpha R\{(U+\bar{c})(\chi'' - \alpha^2\chi) - U''\chi\}. \tag{5.4}$$

Equation (5.4) for $\chi(y)$ is identical with equation (2.3) for $\phi(y)$, and therefore if the latter has a unique solution for given α and R, the eigen-solutions χ and ϕ must be equal, and so are the eigenvalues c and $-\bar{c}$, thus giving that $c_r = 0$. But so far nothing is known about the uniqueness of this eigenvalue problem.

One of the essential weaknesses in applying the usual procedure of hydrodynamic stability theory to low Reynolds number flows seems to lie in its fundamental assumption that the undisturbed flow is, at least approximately, parallel. This assumption is generally satisfied when Reynolds number of the flow is sufficiently large, but it becomes difficult at small Reynolds numbers to maintain parallel flows unless some body force is applied from outside. Therefore, without such external forces, the results obtained by treating the flows as essentially parallel become unrealistic for small Reynolds numbers, and they should only be accepted with these reservations in mind.

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